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SEVERAL NOTES ON CONJECTURES OF BARICZ

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ABSTRACT. In this paper, we mainly give a kind of method to deal with Baricz's conjecture, and establish some new inequalities. we also prove some facts about the functions $g_n(c) = \frac{(c-a)_n}{(c)_n}$ and $f_n(a,c) = \frac{(a)_n(c-a)_n}{(c)_n}$.

1. INTRODUCTION

The following beautiful inequality for Legendre polynomial is due to P. Turán[25]

$$[P_{n+1}(r)]^2 > P_n(r)P_{n+2}(r) \tag{1.1}$$

for all $r \in (-1,1)$ and $n = 0, 1, 2\cdots$, where P_n is the Legendre polynomial, that is,

$$P_n(r) = \frac{d^n}{dr^n}\left[\frac{\left(r^2-1\right)^n}{n!2^n}\right] = {}_2F_1\left(-n, n+1; 1; \frac{1-r}{2}\right). \tag{1.2}$$

Here, ${}_2F_1(a,b;c;r)$ denotes the Gauss hypergeometric function[4] which for given complex numbers a, b and c with $c \neq 0, -1, -2, \cdots$, has the infinite series representation

$$F(a,b;c;r) = {}_2F_1(a,b;c;r) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)}\frac{r^n}{n!}, |r| < 1, \tag{1.3}$$

where$(a)_0 = 1$ and $(a,n) = \prod_{k=0}^{n-1}(a+k)$ is shifted factorial function or the Appell symbol. Later, this classical inequality has been extended in several directions: ultraspherical, Laguerre and Hermite polynomial[22], Jacobi polynomial[16, 17], general class of polynomial[15], Bessel function of the first kind[23], modified Bessel functions of the first kind[8, 20, 24], general Bessel functions[11], generalized trigonometric and hyperbolic functions[10], hypergeometric function[6], generalized complete elliptic integrals[5], regular coulomb wave functions[9], and so on. It is worth mentioning that the inequality (1.1) was improved by Constantinescu[14] and Alzer et al in [2].

Let us consider the notation $F_a(r) = F(a, c-a; c; r)$ where $r \in (0,1)$. In [6], Baricz proposed the following conjecture for Gauss hypergeometric function:

Conjecture 1[6, Open problem] If m_1, m_2 are two-variable means, i. e., for $i = 1, 2$ and for all $x, y, \alpha > 0$, we have

$$m_i(x,y) = m_i(y,x), m_i(x,x) = x, m_i(\alpha x, \alpha y) = \alpha m_i(x,y)$$

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and $x < m_i(x,y) < y$ whenever $x < y$, then find conditions on $a_1, a_2 > 0$ and $c > 0$ for which the inequality

$$m_1\left(F_{a_1}(r), F_{a_2}(r)\right) \le (\ge) F_{m_2(a_1,a_2)}(r) \tag{1.4}$$

holds true for all $r \in (0,1)$.

In the same paper, Baricz also proposed following other open problems.

Conjecture 2[6, Open problems a] For each $n \ge 1$ and $0 < a < c$, the function $g_n(c) = \frac{(c-a)_n}{(c)_n}$ is strictly concave.

Conjecture 3[6, Open problems b] Is it true that for each $n \ge 1$ and $0 < a < c$, the function $f_n(a,c) = \frac{(a)_n(c-a)_n}{(c)_n}$ is strictly concave as a function of two variable?

The main object of this paper is to give a partial solution on Conjecture 1. On the other hand, we also give some properties of the functions $g_n(c) = \frac{(c-a)_n}{(c)_n}$ and $f_n(a,c) = \frac{(a)_n(c-a)_n}{(c)_n}$.

For two distinct positive real numbers x and y, the Arithmetic mean, Geometric mean, Identric mean, Stolarsky mean and Extended mean are respectively defined by

$$A(x,y) = \frac{x+y}{2}, \quad G(x,y) = \sqrt{xy},$$
$$L(x,y) = \frac{x-y}{\log(x) - \log(y)}, \quad x \ne y,$$
$$H(x,y) = \frac{1}{A(1/x, 1/y)},$$
$$I(x,y) = \frac{1}{e}\left(\frac{x^x}{y^y}\right)^{1/(x-y)}, \quad x \ne y,$$
$$U_p(x,y) = \begin{cases} \left(\frac{y^{p+1}-x^{p+1}}{(p+1)(y-x)}\right)^{1/p}, & x \ne y, p \ne -1, 0, \\ y\ , \quad x = y, \end{cases},$$

and

$$E(p,q;x,y) = \left(\frac{p}{q}\frac{y^q - x^q}{y^p - x^p}\right)^{1/(q-p)}, pq(p-q)(x-y) \ne 0,$$

It is easy to see that

$$U_{-1}(x,y) = \lim_{p\to -1} U_p(x,y) = L(x,y), U_1(x,y) = A(x,y),$$
$$U_0(x,y) = \lim_{p\to 0} U_p(x,y) = I(x,y), U_{-2}(x,y) = G(x,y),$$

and

$$E(0,1;x,y) = L(x,y), E(1,2;x,y) = A(x,y),$$
$$E(1,1;x,y) = I(x,y), E(0,0;x,y) = G(x,y).$$

Let $f : I \to (0,\infty)$ be continuous, where I is a sub-interval of $(0,\infty)$. Let M and N be the means defined above, the we call that the function f is MN-convex (concave) if

$$f(M(x,y)) \le (\ge) N(f(x), f(y)) \tag{1.5}$$

for all $x, y \in I$. Recently, generalized convexity/concavity with respect to general mean values has been studied by Anderson et al. in [3]. In [7], Baricz studied that if the functions f is differentiable, then it is (a,b)-convex (concave) on I if and only if $x^{1-a}f'(x)/f(x)^{1-b}$ is increasing (decreasing). It can be observed easily that (1, 1)-convexity means the AA-convexity, $(1,0)$-convexity means the AG-convexity, and $(0,0)$-convexity means GG-convexity.

In [12, 13], the following inequalities were studied:

Theorem 1.1 (Theorem 1[12]). *Let $f : I \to (0,\infty)$ and $I \subseteq (0,\infty)$. Then*
(1) f is LL-convex(concave) if f is increasing and log-convex(concave),
(2) f is AL-convex(concave) if f is increasing and log-convex(concave).

Theorem 1.2 (Theorem 1.6[13]). *Let $f : I \to (0,\infty)$ and $I \subseteq (0,\infty)$. Then the following inequality holds true:*

$$I(f(x), f(y)) \geq f(I(x,y)),$$

$$(I(f(x), f(y)) \leq f(A(x,y))),$$

if the function $f(x)$ is a continuously differentiable, increasing and log-convex(concave).

2. Lemmas

Lemma 2.1. [21] *Let $f, g : [a,b] \to R$ be integrable functions, both increasing or both decreasing. Furthermore, let $p : [a,b] \to R$ be a positive, integrable function. Then*

$$\int_a^b p(x)f(x)dx \cdot \int_a^b p(x)g(x)dx \leq \int_a^b p(x)dx \cdot \int_a^b p(x)f(x)g(x)dx. \tag{2.1}$$

If one of the functions f or g is non-increasing and the other non-decreasing, then the inequality in (2.1) is reversed.

Lemma 2.2. [19] *For $a \leq y < x \leq b$, Then*

$$\varphi\left(\frac{x+y}{2}\right) \leq \frac{1}{x-y}\int_y^x \varphi(u)du \leq \frac{\varphi(x)+\varphi(y)}{2} \tag{2.2}$$

if the function $\varphi(x)$ is convex on $[a,b]$.

Lemma 2.3. [19] *The function $p \mapsto U_p(x,y)$ is strictly increasing on $\mathbb{R} - \{0,-1\}$.*

Lemma 2.4. [19] *The function $p, q, x, y \mapsto E(p,q;x,y)$ is strictly increasing on $pq(p-q)(x-y) \neq 0$.*

Lemma 2.5. [6] *For $0 < a < c \leq 1$ and $r \in (0,1)$, the function $F_a(r) = F(a, c-a; c; r)$ is strictly sub-additive and strictly concave, consequently is strictly log-concave.*

Lemma 2.6. [6] *Let us consider the notation $f_n(a) = (a)_n(c-a)_n$ where $0 < a < c \leq 1$ and $n = 1, 2 \cdots$. Then $f_n(a)$ is strictly concave on $(0,c)$.*

3. Main results

Theorem 3.1. *Let $f : I \to (0,\infty)$ and $I \subseteq (0,\infty)$. For $p < 0$, then the following inequality holds true:*

$$U_p(f(x), f(y)) \geq f(A(x,y)), \tag{3.1}$$

if the function $f(x)$ is a twice differentiable, increasing and concave.

Proof. Easy computation yields

$$(f^p(x))' = pf^{p-1}(x)f'(x),$$
$$(f^p(x))'' = p(p-1)f^{p-2}(x)\left(f'(x)\right)^2 + pf^{p-1}(x)f''(x).$$

Since the function $f(x)$ is increasing and concave, we have

$$(f^p(x))' \leq 0, (f^p(x))'' \geq 0.$$

So the function $f^p(x)$ is decreasing and convex on $p < 0$. By using Lemma 2.1, we have

$$\int_x^y f^p(u)du \int_x^y f'(u)du \leq \int_x^y 1du \int_x^y f^p(u)f'(u)du. \tag{3.2}$$

On the other hand, simple computation and substitution $t = f(u)$ yield

$$\ln U_p(f(x), f(y)) = \frac{1}{p}\ln\left(\frac{f^{p+1}(y)-f^{p+1}(x)}{(p+1)(f(y)-f(x))}\right)$$
$$= \frac{1}{p}\ln\left(\frac{\int_x^y f^p(u)f'(u)du}{\int_x^y f'(u)du}\right)$$

Considering inequality (3.2) and Lemma 2.2, we have

$$\ln U_p(f(x), f(y)) \geq \frac{1}{p}\ln\left(\frac{\int_x^y f^p(u)du}{\int_x^y 1du}\right)$$
$$\geq \frac{1}{p}\ln\left(f^p(A(x,y)\right) = \ln f(A(x,y)).$$

The proof is complete. □

It is easy to verify that the function $F_a(r)$ satisfies all conditions of Theorem 3.1 by using Lemma 2.5. So, we obtain following Corollary 3.1:

Corollary 3.1. *For* $0 < a_i < c \leq 1, i = 1, 2, r \in (0,1)$ *and* $p < 0$, *then*

$$U_p\left(F_{a_1}(r), F_{a_2}(r)\right) \geq F_{A(a_1,a_2)}(r). \tag{3.3}$$

Similarly, applying results of Theorem 1.1 and 1.2, the following theorem holds true:

Theorem 3.2. *For* $0 < a_i < c \leq 1, i = 1, 2, r \in (0,1)$, *then*
(1)$L\left(F_{a_1}(r), F_{a_2}(r)\right) \leq F_{L(a_1,a_2)}(r)$;
(2)$L\left(F_{a_1}(r), F_{a_2}(r)\right) \leq F_{A(a_1,a_2)}(r)$;
(3)$I\left(F_{a_1}(r), F_{a_2}(r)\right) \leq F_{A(a_1,a_2)}(r)$.

Remark 3.1. Corollary 3.1 and Theorem 3.2 give a partial solution of Conjecture 1. It is worth noting that our method offers a kind of way to solve conjecture of Baricz. Similar idea may apply other means. we give two meaning results.

Theorem 3.3. *For* $p \geq 1, p' \leq 1$, *Let* $f : I \to (0,\infty)$ *and* $I \subseteq (0,\infty)$. *Then the following inequality holds true:*

$$U_p(f(x), f(y)) \geq f(U_{p'}(x,y)), \tag{3.4}$$

if the function $f(x)$ *is a twice differentiable, increasing and convex.*

Proof. Because the function $f(x)$ is increasing and convex, we have

$$(f^p(x))' \geq 0, (f^p(x))'' \geq 0,$$

and consequently $f^p(x)$ is increasing and convex. By using Lemma 2.1, we also have

$$\int_x^y f^p(u)du \int_x^y f'(u)du \le \int_x^y 1du \int_x^y f^p(u)f'(u)du. \qquad (3.5)$$

So, simple computation and substitution $t = f(u)$ yield

$$\begin{aligned}&\ln U_p(f(x), f(y)) = \tfrac{1}{p}\ln\left(\tfrac{f^{p+1}(y)-f^{p+1}(x)}{(p+1)(f(y)-f(x))}\right)\\ &= \tfrac{1}{p}\ln\left(\tfrac{\int_x^y f^p(u)f'(u)du}{\int_x^y f'(u)du}\right) \ge \tfrac{1}{p}\ln\left(\tfrac{\int_x^y f^p(u)du}{\int_x^y 1du}\right)\\ &\ge \tfrac{1}{p}\ln\left(f^p(A(x,y)\right) = \ln f(A(x,y))\end{aligned}$$

by using Lemma 2.2. Considering increasing property of the function $f(x)$ and Lemma 2.3, we have

$$U_p(f(x), f(y)) \ge f(U_{p'}(x, y)).$$

□

Theorem 3.4. *Let $f : I \to (0,\infty)$ and $I \subseteq (0,\infty)$.*
(1) For $p \le 1, q \le 2, q-p \ge 1$, we have

$$E(p, q; f(x), f(y)) \ge f(E(p, q; x, y)) \qquad (3.6)$$

If the function $f(x)$ is a twice differentiable, increasing, convex and the another function $f^{p-1}(x)f'(x)$ is also increasing.
(2) For $q-p \ge 1$, we have

$$E(p, q; f(x), f(y)) \ge f(A(x, y)) \qquad (3.7)$$

If the function $f(x)$ is a twice differentiable, increasing, convex and the other function $f^{p-1}(x)f'(x)$ is also increasing.

Proof. Since the proof of part (2) is similar to part (1), we only prove the part (1) here. Similar to the proof of Theorem 3.1, we easily know that the function $f^{q-p}(x)$ is increasing and convex. An easy computation and substitution $t = f(u)$ yield

$$\begin{aligned}&\ln E(p, q; f(x), f(y))\\ &= \tfrac{1}{q-p}\ln\left(\tfrac{p}{q}\tfrac{f^q(y)-f^q(x)}{f^p(y)-f^p(x)}\right)\\ &= \tfrac{1}{q-p}\ln\left(\tfrac{\int_x^y f^{q-1}(u)f'(u)du}{\int_x^y f^{p-1}(u)f'(u)du}\right).\end{aligned}$$

Since the functions $f^{q-p}(x)$ and $f^{p-1}(x)f'(x)$ are increasing on $I \subseteq (0,\infty)$, now by using 2.1 again, we have

$$\int_x^y f^{p-1}(u)f'(u)du \int_x^y f^{q-p}(u)du \le \int_x^y 1du \int_x^y f^{q-1}(u)f'(u)du. \qquad (3.8)$$

So, we have

$$\begin{aligned}&\ln E(p, q; f(x), f(y)) \ge \tfrac{1}{q-p}\ln\left(\tfrac{\int_x^y f^{q-p}(u)du}{\int_x^y 1du}\right)\\ &\ge \tfrac{1}{q-p}\ln\left(f(A(x,y))\right)^{q-p} = \ln f(E(1,2;x,y))\\ &\ge \ln f(E(p,q;x,y))\end{aligned}$$

by using Lemma 2.2 and Lemma 2.4. This completes the proof. □

Next, we shall give some properties of the functions $g_n(c)$ and $f_n(a, c)$.

Theorem 3.5. *For each $n \ge 1$ and $0 < a < c$, the function $g_n(c) = \frac{(c-a)_n}{(c)_n}$ is strictly increasing and strictly log-concave on (a,∞).*

Proof. After some elementary computation, we get

$$g_n'(c) = a g_n(c) \sum_{k=0}^{n-1} \frac{1}{(c+k)(c+k-a)}$$

and

$$\left(\frac{g_n'(c)}{g_n(c)}\right)' = \sum_{k=0}^{n-1} \left(\frac{1}{(c+k)^2} - \frac{1}{(c+k-a)^2}\right)$$

which implies that $g_n'(c) > 0$ and $\left(\frac{g_n'(c)}{g_n(c)}\right)' < 0$. This finishes the proof. □

Theorem 3.6. *For $0 < a_i < c \le 1, i = 1, 2$, the following inequality holds true:*

$$\left(f_n\left(\frac{a_1+a_2}{2}, \frac{c_1+c_2}{2}\right)\right)^4 \ge f_n(a_1, c_1) f_n(a_1, c_2) f_n(a_2, c_1) f_n(a_2, c_2). \tag{3.9}$$

Proof. From Lemma 2.6, we know that the function $f_n(a)$ is strictly log-concave. Combining with log-concave of $g_n(c)$ in Theorem 3.5, we have

$$\begin{aligned} f_n\left(\tfrac{a_1+a_2}{2}, \tfrac{c_1+c_2}{2}\right) &= \frac{\left(\frac{a_1+a_2}{2}\right)_n \left(\frac{c_1+c_2}{2} - \frac{a_1+a_2}{2}\right)_n}{\left(\frac{c_1+c_2}{2}\right)_n} \\ &\ge \frac{1}{\left(\frac{c_1+c_2}{2}\right)_n} \sqrt{(a_1)_n \left(\tfrac{c_1+c_2}{2} - a_1\right)_n (a_2)_n \left(\tfrac{c_1+c_2}{2} - a_2\right)_n} \\ &= \sqrt{(a_1)_n (a_2)_n} \sqrt{\frac{\left(\frac{c_1+c_2}{2} - a_1\right)_n}{\left(\frac{c_1+c_2}{2}\right)_n} \frac{\left(\frac{c_1+c_2}{2} - a_2\right)_n}{\left(\frac{c_1+c_2}{2}\right)_n}} \\ &\ge \sqrt{(a_1)_n (a_2)_n} \sqrt[4]{\frac{(c_1-a_1)_n}{(c_1)_n} \frac{(c_2-a_1)_n}{(c_2)_n} \frac{(c_1-a_2)_n}{(c_1)_n} \frac{(c_2-a_2)_n}{(c_2)_n}} \\ &= \sqrt[4]{f_n(a_1, c_1) f_n(a_1, c_2) f_n(a_2, c_1) f_n(a_2, c_2)}. \end{aligned}$$

This finishes the proof. □

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